Estimation of magnetotelluric transfer functions from radio transmitters

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ABSTRACT

A new method to estimate magnetotelluric (MT) transfer functions using radio transmitters as the source has been developed using an extended version of Weidelt’s C function expansion. The expansion coefficients are found from solving a mixed determined linear inverse problem by a truncated singular-value decomposition technique with a truncation level determined automatically from the data. Synthetic tests with both noise-free and noisy data show the new method is accurate and robust with respect to lack of transmitters in certain bands. A test profile with 42 stations from a dump-site investigation in the Netherlands shows that the estimated transfer functions are remarkably stable and smooth as a function of frequency and distance along the profile. Compared with the standard band-averaging technique, even the formal error bars are generally smaller and more realistic with the new method because the bias introduced by assuming that transfer functions are constant in a small subband is removed.

INTRODUCTION

Background

The estimation of transfer functions in standard magnetotelluric (MT) measurements has attracted a great deal of attention over the past 30 years (see, for example, Chave et al., 1987; Larsen, 1989; Egbert and Livelybrooks, 1996; Banks, 1998; Ritter et al., 1998). The transfer functions can be used to estimate the electrical properties of the earth’s crust and mantle. Assuming a plane-wave excitation, we can show that, for a fixed position in space at a given frequency, there exist unique transfer functions between the two horizontal magnetic-field components \( H_x \) and \( H_y \) and the two horizontal electric-field components \( E_x \) and \( E_y \) [and the vertical magnetic-field component \( H_z \)]. In practice a number of problems are related to estimating these transfer functions. The electromagnetic (EM) components measured at the earth’s surface may become noisy from various sources: internal electronic noise from the instrumentation, wind noise especially affecting the magnetic field measurements, manmade noise from imbalanced power consumption, electrical railways with dc currents entering grounding points that change with the position of the trains, dc power transmission in submarine cables, etc. The basic assumption of a plane-wave source field may also be violated at periods for which the inductive scale length is comparable to the distance to the sources, which could be in the ionosphere or in tropical regions where lightning is predominant. The assumption is also violated if the receiver is too close to a particular source, e.g., a VLF radio transmitter.

Estimating normal MT transfer functions

The MT transfer functions for plane-wave excitation are defined by

\[
\begin{align*}
E_x & = Z_{xx} H_x + Z_{xy} H_y, \\
E_y & = Z_{yx} H_x + Z_{yy} H_y, \\
H_z & = T_{zx} H_x + T_{zy} H_y.
\end{align*}
\]

The impedance tensor

\[
Z = \begin{pmatrix}
Z_{xx} & Z_{xy} \\
Z_{yx} & Z_{yy}
\end{pmatrix}
\]

with its four complex components completely describes the connection between horizontal magnetic \( H_\phi = (H_x, H_y) \) and electric \( E = (E_x, E_y) \) fields at a given frequency. The geomagnetic transfer function (tipper vector) \( \mathbf{T} = (T_{zx}, T_{zy}) \), with its two complex components, completely describes the connection between the horizontal magnetic components and the vertical magnetic-field component \( H_z \).

The classical way of estimating the impedance tensor and the tipper vector is to assume that the noise is concentrated in the electric-field components (Pedersen, 1982). If the noise in the magnetic channels is negligible, then the estimated impedance tensor and tipper vector, \( \hat{Z} \) and \( \hat{\mathbf{T}} \), are bias free even with very loose constraints on the statistics of the noise in the electrical channels. Minimizing the residual power in
the electric-field \(|E - \mathbf{Z} \cdot \mathbf{H}|\) and the vertical magnetic-field \(|\mathbf{H} \cdot \mathbf{T} \cdot \mathbf{H}|\) components with respect to the unknown transfer functions provides least-squares normal equations that can be solved directly. With Gaussian statistics, confidence limits can be assigned to the estimates, and with non-Gaussian statistics robust methods can be used (Larsen, 1989) to identify those parts of the data that deviate from the Gaussian behavior and to downweight or neglect those parts in the estimation procedure.

The remote reference technique was introduced to remove the need to assume noise-free magnetic fields (Gamble et al., 1979). Using the magnetic field at a site remote from the site under investigation to predict the magnetic field there makes it possible to obtain unbiased estimates of all transfer functions. The estimates are not biased if the noise is not correlated. To some extent, departures from plane-wave behavior are propagated into noise in the electric-field components. In the case of intermediate fields, remote reference and/or robust methods can be used to remove partly the effect of nonplane-wave source fields on the transfer functions.

Constant transfer function assumption

A fundamental assumption in standard MT transfer function estimation schemes is that the transfer functions are constant in the narrow band in which they are estimated. Typically, a bandwidth of half an octave is used. As we shall see later, this assumption may distort the estimates that may easily exceed the level of acceptable errors in radio–MT applications.

Consider a simple 1-D model, a homogeneous half-space, for which the \(x\)-component of the electric field and the \(y\)-component of the magnetic field are measured. The impedance for a homogeneous half-space is given by

\[
Z_{xy}(\omega) = i\omega(\omega\lambda_0\sigma)^{-1/2} = \sqrt{i\lambda_0/\sigma}\sqrt{\omega},
\]

where \(\omega\) is the cyclic frequency, \(\mu_0\) is the vacuum permeability, and \(\sigma\) is the conductivity. The impedance varies as the square root of \(\omega\), meaning that over a bandwidth of half an octave \(|Z_{xy}|\) varies by a factor of \(\sqrt{2} \approx 1.19\). This corresponds to a possible bias in apparent resistivity of about 20% from the true value in the middle of the band. In normal MT cases since the signal is evenly distributed over the band, this is not a serious problem. However, the unevenly distributed signal used in the radio–MT method may cause considerable bias effects that widely supersede the bias effect attributable to noise in the magnetic channel. (Typically, magnetic S/N ratios are 30 dB.) To further complicate matters, in real situations it may easily happen that the frequency separation between discrete radio transmitters exceeds one octave. Clearly there is a need for a new way of estimating planewave transfer functions.

Radio–MT source signals

The distribution of radio transmitters deviates considerably from region to region worldwide. In Sweden and the Netherlands, the number of available radio transmitters is nearly the same. As an example, in Figure 1 we show the distribution of radio transmitter frequencies in the Netherlands during mid-November 1998. Forty transmitter frequencies could be detected lying at least 14 dB above the background horizontal magnetic-field noise. The frequency resolution is 122 Hz, corresponding to 16 384 samples and a sampling rate of 2 MHz. The distribution of selected frequencies is highly irregular, with a concentration in the VLF band around 16 kHz and above 40 kHz to 250 kHz. There is a transmitter gap in the octave from 20 to 40 kHz. Another complication is that even though the number of transmitters in a given half-octave band is high, their azimuths (the direction between the receiving and transmitting sites measured from geomagnetic north) may deviate only slightly. This also may cause the least-squares normal equations to become nearly singular and the estimates highly unstable. Also, in the given example the various transmitters have different S/N ratios that vary between 14 and 40 dB.

In this work, we first explain a method introduced by Weidelt (1972) to overcome the problems with irregularities in 1-D normal MT data by finding a smooth transfer function using an expansion of the MT transfer functions. Then a new approach is introduced to extend the method to 3-D MT data, with emphasis on solving irregularities found in radio–MT data. The method is then examined using a set of synthetically generated data. Finally, the application of the method to real radio–MT data is represented.

**METHOD**

**1-D representation of MT transfer functions**

A plane-wave source field exciting a horizontally stratified earth model can be characterized by the inductive scale length \(C\) (Weidelt, 1972; Parker and Booker, 1996), which in turn is related to the impedance tensor element \(Z_{xy}\) as

\[
Z_{xy}(\omega) = i\omega C(\omega),
\]

where \(Z_{xy}\) can be regarded as a filter relating the output \(E_x\) to the input \(H_z\). Weidelt (1972) shows that \(C\) is a minimum phase filter and establishes a dispersion relation between the phase and absolute value of \(C\). He also quotes the following expansion of \(C\) in terms of elementary fractions:

\[
C(\omega) = \frac{\sigma_0}{\lambda} + \sum \frac{a_{m}}{\lambda_m + i\omega}; \lambda_m > 0; a_m' > 0.
\]

![Fig. 1. An example of radio transmitter distribution in Collendoorn, the Netherlands](image)
Looking at the inductive scale length defined by equation (4) as a filter allows for an interpretation of its inverse Fourier transform as an impulse response function. In this case the impulse response function is particularly simple—namely, a superposition of exponential functions with decay constants $\lambda_m$ weighted by the amplitude $a_m$. Since both $\lambda_m$ and $a_m$ are positive, the exponential basis functions decrease steadily to zero as a function of time.

For a set of $N$ data points, neglecting the first term (since there is no dc term in the radio-MT method) and introducing the data period $2\pi/\omega = T = T_n$, $n = 1, \ldots, N$ and the model period $T_m$, $m = 1, \ldots, M$, we find for the 1-D impedance

$$Z_{1D}(T_n) = \sum_{m=1}^{M} \frac{i a_m}{T_n + iT_m}; T_n > 0; a_m > 0. \quad (5)$$

Larsen (1975) uses a polynomial form to represent the elements of the impedance tensor. He uses $Z_p$, as an exact response function belonging to a preliminary estimate of the conductivity model. Later, to construct a smooth transfer function, Larsen et al. (1996) use a formulation similar to equation (5) to generate a 1-D casual minimum-phase transfer function $Z^{\ast}_{1D}(\omega)$.

As discussed by Parker and Booker (1996), this form may also be used to check if transfer functions satisfy 1-D conditions. If they do not, error bars may be unrealistically small or bias effects may be big or even the conductivity structure may be two or three dimensional. Thus, in 1-D conditions the expansion can be used to identify bad data parts.

**Extended parameterization of MT transfer functions.**—How can we extend the expansion representation of the off-diagonal elements used for a 1-D earth structure to describe an arbitrarily multidimensional impedance tensor and tipper vectors? First, even though off-diagonal impedance elements seem to satisfy 1-D conditions, i.e., $Z_{xy} = -Z_{yx}$, to within the accuracy obtained in most MT experiments, it is clear that the diagonal elements do not show such a simple behavior, i.e., $Z_{xx} \neq Z_{yy}$, $Z_{xy} \neq 0$. It is well known that static shift effects for a regional 2-D structure show up as leakage from off-diagonal elements into diagonal elements of the same column. In the regional coordinate system (coordinate axes parallel and perpendicular to the regional strike), the diagonal elements thus depend upon experience and empirical evidence, it is not so common to find off-diagonal impedance elements that deviate significantly from 1-D behavior to within the accuracy of the measurements.

Second, plane-wave transfer functions are smooth functions of frequency. Exactly how smooth it is difficult to say in the general case. Only for one dimension and the TM mode in two dimensions is the simple representation for $C(\omega)$ strictly valid (Weidelt, 1972; Weidelt and Kaikkonen, 1994). A simple extension of equation (4) is to allow the coefficient $a_m$ to take on any real value. In this case the impulse response functions will no longer be steadily decaying as a function of time. In some time intervals the function may increase and may even change sign. With this extended but less restrictive parametric form, there will still be a coupling between the real and the imaginary parts of the impedance tensor elements and the tipper vector elements.

There are other methods to generate smooth transfer functions for 2-D and 3-D data (Larsen, 1989; Larsen et al., 1996). As an example, using a smooth correction function $U(\omega)$ and distortion function $D(\omega)$, Larsen et al. (1996) generate a general form of smooth transfer function defined as $Z(\omega) = U(\omega)D(\omega)Z^{\ast}_{1D}(\omega)$. The values $U(\omega)$ and $D(\omega)$ are represented by two different $N$ Chebyshev polynomials. As mentioned before, $Z^{\ast}_{1D}(\omega)$ is estimated by the partial fraction expansion, and the real positive $a_m$ and $\lambda_m$ are calculated by the $D^+$ routine (Parker, 1980). They use an iterative procedure to find the best transfer function. The iteration stops when the number of new identified outliers is <0.5% of the total number of data and $U(\omega)$ is very close to one.

**Formulation of the inverse problem with extended parameterization.**—Let each of the elements $Z_{xx}$ and $Z_{yx}$ be represented by the expansion of equation (5). Then, using equation (1),

$$E_x(T_n) = H_x(T_n) \sum_{m=1}^{M} \frac{i a^x_m}{T_n + iT_m}$$

$$+ H_y(T_n) \sum_{m=1}^{M} \frac{i a^y_m}{T_n + iT_m}; n = 1, \ldots, N, \quad (6)$$

where $N$ is the number of data periods and $M$ is the number of model periods or the number of coefficients are to be estimated for each impedance tensor element. This can be further simplified to

$$E_x(T_n) = \sum_{m=1}^{M} G^x_{nm} a^x_m + \sum_{m=1}^{M} G^y_{nm} a^y_m, \quad (7)$$

with

$$G^x_{nm} = H_x(T_n) \frac{i}{T_n + iT_m} \quad \text{and} \quad G^y_{nm} = H_y(T_n) \frac{i}{T_n + iT_m}. \quad (8)$$

Written in full using vector and matrix notation,

$$E_x = G^x a^x + G^y a^y = Gm,$$  

(9)

where $G = [G^x | G^y]$ and $m = \begin{pmatrix} a^x \\ a^y \end{pmatrix}$.

The vector $E_x$ is an $N \times 1$ data vector that contains a measured electric field in the $x$-direction, $G$ is an $N \times 2M$ data kernel matrix, and $m$ is a $2M \times 1$ model parameter vector that contains the expansion coefficients.

The data kernels can also be written as

$$G^x_{nm} = \frac{i H_x(T_n)}{T_n + iT_m}, \quad G^y_{nm} = \frac{i H_y(T_n)}{T_n + iT_m}. \quad (10)$$
or can be written in vector/matrix notation as

\[ \mathbf{G} = [\mathbf{G}^T \mid \mathbf{G}^*] = \mathbf{DW}, \]  

(11)

in which

\[ \mathbf{D} = [\text{diag}(\mathbf{H}_x)\mid \text{diag}(\mathbf{H}_y)] = \begin{bmatrix} H_x^1 & 0 & H_y^1 & 0 \\ \vdots & \cdots & \vdots & \cdots \\ H_x^N & 0 & H_y^N \end{bmatrix} \]

(12)

is an \( N \times 2N \) matrix and

\[ \mathbf{W} = \begin{bmatrix} w_{11} \cdots w_{1M} \\ \vdots \\ w_{n1} \cdots w_{nM} \\ w_{N1} \cdots w_{NM} \end{bmatrix}, \quad \mathbf{u}_m = \frac{i}{T_n + iT_m} \]

(13)

is a \( 2N \times 2M \) matrix. The values \( H_x^r \) and \( H_y^r \) are the measured horizontal magnetic-field components at data period \( T_n \).

We let the set of model periods \( T_n \) be identical for all impedance elements. In practice we choose two to four periods per octave such that the minimum and maximum model periods lie well beyond the minimum and maximum data periods. We seek a solution \( \mathbf{m}^{est} \) such that the weighted prediction error energy is minimized. When requiring that the solution be real, it can be written formally as

\[ \mathbf{m}^{est} = [\text{Re}(\mathbf{GG})]^{-1}[\text{Re}(\mathbf{g}))], \]

(14)

where the tilde indicates the conjugate transpose of the matrix. The data vector \( \mathbf{d} \) is given by

\[ \mathbf{d} = \{ E_x(T_n) \}. \]

(15)

In practice, because of random phase of the source (in our case, radio transmitters), the spectral matrix is stored as the stacked auto- and crosstrokes of electric- and magnetic-field components. Hence, we rewrite the terms in equation (14) as

\[ \tilde{\mathbf{G}} = \tilde{\mathbf{W}} \mathbf{D} \mathbf{D} \tilde{\mathbf{W}} = \tilde{\mathbf{W}} \left[ \begin{array}{ll} \text{diag}(\mathbf{H}_x) & \text{diag}(\mathbf{H}_y) \\ \text{diag}(\mathbf{H}_x) & \text{diag}(\mathbf{H}_y) \end{array} \right] \mathbf{W} \]

(16)

and

\[ \tilde{\mathbf{G}} \mathbf{d} = \tilde{\mathbf{W}} \mathbf{D} \mathbf{E}_x = \tilde{\mathbf{W}} \left( \begin{array}{cc} \mathbf{S}_{HE_x} & \mathbf{S}_{HE_y} \end{array} \right). \]

(17)

where

\[ \begin{bmatrix} (H_x^1)^* H_x^1 & 0 \\ \vdots & \cdots \\ (H_x^N)^* H_x^N \end{bmatrix}, \quad \begin{bmatrix} (H_y^1)^* H_y^1 & 0 \\ \vdots & \cdots \\ (H_y^N)^* H_y^N \end{bmatrix} \]

(18)

and * means the complex conjugate.

Equations (16) and (17) allow for a simpler representation of the inverse problem in terms of the auto- and crosstrokes (in the radio–MT case at the selected transmitter frequencies), denoted by vectors \( \mathbf{S}_{HE_x}, \mathbf{S}_{HE_y}, \mathbf{S}_{HE_y}, \mathbf{S}_{HE_x}, \mathbf{S}_{HE_y}, \) and \( \mathbf{S}_{HE_y} \).

Taking into account that different transmitters have different SIN ratios, we can reformulate the inverse problem as one for which a weighted sum of squares of the residuals is minimized. This can be accomplished by scaling \( \mathbf{W} \) with the corresponding standard deviations of the output channels (\( E_x, E_y, H_x \)). The standard deviations of the data, i.e., the background noise level, can be obtained directly from the measurements by running a median filter over the entire measuring band.

Transfer function estimates and their covariance and resolution matrices.—The estimate \( \mathbf{m}^{est} \) can be transformed to impedance estimates using equation (5) at frequencies for which transmitters exist. The model covariance matrix, for example, for the first row of impedance tensor is

\[ [\text{cov}(\mathbf{Z}_x^{est})] = [\mathbf{W}][\text{cov}(\mathbf{m}^{est})][\mathbf{W}] \]

(19)

and

\[ [\text{cov}(\mathbf{m}^{est})] = [\text{Re}(\mathbf{GG})]^{-1}[\mathbf{G}^T][\text{cov}(\mathbf{d})][\mathbf{G}], \quad [\text{Re}(\mathbf{GG})]^{-1}, \]

(20)

where the covariance of the real part is equal to the covariance of the imaginary part.

Assuming an uncorrelated data, the cross-covariances can be neglected. This means that data covariance matrices are diagonal and can be expressed as

\[ [\text{cov}(\mathbf{d})] = [\text{cov}(\mathbf{d})] + [\text{cov}(\mathbf{d})] = 2[\text{cov}(\mathbf{d})] \]

(21)

where the last term is the noise in the \( x \)-direction of the electric field. If can easily be found as the background noise level of the measurements. The resolution matrix of the impedance in the \( x \)-direction is defined as

\[ \mathbf{Z}_x^{est} = \mathbf{R} \mathbf{Z}_x, \]

(22)

where \( \mathbf{Z}_x \) defines the true value of the known impedance element. But \( \mathbf{Z}_x^{est} = \mathbf{W} \mathbf{m}^{est} = \mathbf{WR}_m \mathbf{m} = \mathbf{WR}_m \mathbf{W}^{-1} \mathbf{mW} = \mathbf{WR}_m \mathbf{W}^{-1} \mathbf{Z}_x \), where \( \mathbf{W}^{-1} \) is the generalized inverse of \( \mathbf{W} \), which can easily be calculated once \( \mathbf{W} \) is known, and \( \mathbf{R}_m \) is the resolution matrix of the expansion coefficients. Then the resolution
matrix of the impedance element and the resolution matrix of the expansion coefficients are related by

\[ R_z = W R_m W^{-1} \]  

(23)

Maximum likelihood estimate (weighted least squares).—For each frequency the S/N ratio is known for all output channels, i.e., \( E_x, E_y \), and \( H_z \), respectively. Then an unbiased estimate (assuming as before that the horizontal magnetic field is noise free) can be obtained by the scaled equations:

\[ d' = \text{diag}\{\Delta S_{Ex,Es}\}^{-1} d \]
\[ = \text{diag}\{\Delta S_{Ex,Es}\}^{-1/2} G_m = G'm. \]  

(24)

The solution to this equation is given by

\[ m^{est} = \text{Re}(\tilde{G}'G')^{-1}\text{Re}[\tilde{G}'d'], \]  

(25)

which is similar to equation (14). The covariance of the model parameters for the weighted least squares can be expressed by

\[ \text{[cov}\{m^{est}\}] = \text{Re}(\tilde{G}'G')^{-1}. \]  

(26)

The inverse problem can be solved by a truncated singular-value decomposition (TSVD).

SYNTHETIC DATA TESTS

Noise-free data

To check the method, a set of EM-field components in the frequency domain was generated synthetically. The radiofrequency distribution was chosen to be as close as possible to a real situation observed at the Collendoorn test site, the Netherlands, in mid-November 1998. First we assumed that only the data frequencies that represent a radio transmitter shown in Figure 1 were available for the transfer function estimation. The selected frequencies varied from 14 to 250 kHz. Magnetic fields belonging to adjacent frequencies were assumed to be linearly polarized in perpendicular directions.

We generated a data set that reflects the response of a simple 1-D, three-layer model, with \( Z_{xy}(\omega) = Z_{yx}(\omega) = 0 \) and \( Z_{xx}(\omega) = -Z_{yy}(\omega) \). The model shown in Figure 2a has a resistive overburden, overlying a thick conductive layer, in turn overlying a resistive unit at the bottom. Figures 2b and 2c show the synthetic apparent resistivity and phase for a range that spans the data frequencies.

Then, using equations (1) and the impedance tensor model, we generated the spectral matrix elements for the entire set of data frequencies. The model frequencies, sampled two per octave from 100 Hz to 25 MHz, range almost two decades away from the lowest and highest data frequencies. Because the data are noise free, we used the normal least-squares method and formed the necessary vectors and matrices \( (d, G, m) \). Because of the structure of the first term in equation (14), \( \text{Re}(GG')^{-1} \), we used the TSVD technique to find the inverse. The eigenvalue spectrum of \( \text{Re}(GG') \) is shown in Figure 3. These values give an idea about the degree of singularity, although it is difficult to estimate where to truncate the singular-value expansion.

To show the effect of the number of eigenvectors included on the final estimates, we selected three examples to demonstrate the differences. Figure 4 represents the estimated apparent resistivities and phases (circles) using 8, 20, and 52 eigenvectors. It is clear that the estimates are far from the true model response (plus signs) for 8 and 52 eigenvectors and they match nearly perfectly the model response with 20 eigenvectors. The reason for this behavior is well known in the theory of singular-value decomposition (Menke, 1989).

FIG. 2. (a) Three-layer model, resistive–conductive–resistive, used to generate the synthetic data. (b) Apparent resistivity of the model. (c) Impedance phase of the model.

FIG. 3. The eigenvalue spectrum of the matrix \( \text{Re}(GG') \). In this figure every second eigenvalue is shown.
When the number of the eigenvectors is too small, the estimated model parameters are averaged and the final model is highly smoothed. For a large number of eigenvectors, because of the effect of the small eigenvalues and numerical instability, the estimated direction in the model space is perturbed dramatically and the estimated model response is far from the true response. The test also showed that the model frequency bandwidth does not affect the final estimates if the model bandwidth is at least one decade wider than the data bandwidth on both sides. Increasing the model bandwidth just increases the processing time. Two model frequencies per octave are sufficient for the models tested here. Increasing this number only increases processing time.

We also checked the behavior of the estimated diagonal impedance tensor elements, $Z_{xx}^{est}$ and $Z_{yy}^{est}$. As we expected, when the fit to the true model was the best, these elements were close to zero (on the order of $10^{-3}$).

Noisy data

To simulate the real MT measurements situation even better, a 10% random noise that corresponds to 1% error mapped to the transfer functions was added to the amplitude of the output channels. In real radio–MT measurements, the weights used in least squares are assumed to be those derived from the measured S/N ratios (in this synthetic case, 1% of the autopowers). Using the same resistivity model as before, synthetic data in the form of auto- and crosspower spectra for all the channels were generated. We used a weighted least-squares approach and formed $\mathbf{d}'$ and $\mathbf{G}'$ according to equation (24). Figure 5 shows the maximum likelihood solutions with 8 and 16 eigenvectors included in the calculations. Results are almost the same as the noise-free results except that the fit for 16 eigenvectors is worse compared with the noise-free case. However, the results are sufficiently acceptable to reproduce the real impedance.

Fig. 4. Estimated apparent resistivity and phase curves with noise-free synthetic data: (a) 8, (b) 20, and (c) 52 eigenvectors included.
model. Each figure also shows the estimated relative apparent resistivity error and the absolute phase error. As illustrated, the larger the number of eigenvectors used in the singular-value expansion, the larger the errors. Menke (1989) shows that if the data are uncorrelated with uniform variance $\sigma^2_d$, the covariance matrix of the estimated model parameters is given by

$$[\text{cov}(m^{\text{est}})] = \sigma^2_d V_p \Lambda_p^{-2} V_p^T$$

and the variance of the model parameters can be expressed by

$$\sigma^2_{ii}(p) = \sigma^2_d \sum_{j=1}^{p} v_{ij}^2 \lambda_j^{-2},$$

in which $v_{ij}$ is the $i$th direction cosine of the $j$th principal axis in the model space, $\lambda_j$ is the $j$th eigenvalue, and $p$ is the truncation number. Then the difference between the variances of the two estimated models, one with $p$ and the other one with $p+1$ eigenvectors, can be written as

$$\Delta = \sigma^2_d \sum_{j=1}^{p} v_{ij}^2 \lambda_j^{-2}.$$  

Because $\Delta$ is a positive number, the variance of each model parameter increases with an increasing number of eigenvectors. Especially for small eigenvalues, this increase may become dramatic.

We have addressed the problem of first defining and then finding the optimum number of eigenvectors/eigennumbers. This is the topic of the next section.

Data fit and truncated singular-value decomposition (SVD).—The total misfit between observed data $d_{\text{obs}}^i$ and model or estimated data $d_{\text{est}}^i$ is defined as

$$\Delta = \sigma^2_d \sum_{j=1}^{p+1} v_{ij}^2 \lambda_j^{-2}.$$  

Fig. 5. Estimated apparent resistivity, phase, and their errors in the x-direction of an electric field with 10% random noise added to the data: (a) 8 and (b) 16 eigenvectors included.
\[
\chi^2 = \sum_{i=1}^{N} \frac{(d_{\text{obs}}^i - d_{\text{est}}^i)^2}{\sigma_i^2}.
\]  

(30)

Assume that the estimated model that resembles the true model and fits the data best can be spanned in the model space by the eigenvectors belonging to the \( P_{\text{true}} \) largest eigenvalues. This means that

\[
\chi^2(p) > \chi^2(p + 1) \quad \text{for} \quad p < p_{\text{true}}
\]
\[
\chi^2(p) > \chi^2(p + 1) \quad \text{for} \quad p \geq p_{\text{true}}
\]  

(31)

Consequently, the normalized \( \chi^2 \), defined as

\[
Q(p) = \frac{\chi^2}{2N - P},
\]  

(32)

satisfies

\[
Q(p) < Q(p + 1) \quad \text{for} \quad p \leq p_{\text{true}}.
\]  

(33)

We chose this criterion to stop adding new eigenvectors/eigenvalues to the inverse of the matrix \( [\text{Re} (G G^\dagger)] \). Theoretically, the minimum of \( Q(p) \) found in this way can be a local minimum, but our experience from working with synthetic and real data showed this to rarely be the case. To illustrate the technique, we show in Figure 6 the behavior of \( Q(p) \) for both output channels \( E_x \) and \( E_y \). Because of the random errors introduced in the simulations, the two curves differ slightly, although both show a minimum of about unity at \( p = 16 \). Notice that at \( p = 10, Q(p) \) is practically at its minimum. By increasing \( p \) to 16, only minor improvements in data fit occur. Also, when the inverse of the matrix is formed, the small eigenvalues cause numerical instabilities. To stabilize the procedure, we added a damping factor equal to \( 10^{-10} \) of the largest eigenvalue to the diagonal terms.

With one data segment (one stack), the standard errors on the apparent-resistivity estimates were generally <4%. This means four stacks are necessary to reduce the standard errors to <2%, which would be the bias error if the input channels had an S/N ratio of 20 dB.

**REAL DATA TESTS**

A new system called ENVIRO-MT was developed at the Department of Earth Sciences of Uppsala University and at Metronix GmbH (Germany) from 1997 to 1999. The data used in this paper are part of the data set from the test of the system outside a waste disposal site at Collendoorn, the Netherlands. Collendoorn, a flat area covered by marine Pliocene sediments, has been used as a dumping site since 1945. From a geohydrological point of view, a sequence of permeable and impermeable layers with varying thicknesses is prominent in the area. At the sides of the waste disposal is an impermeable clay layer at an average depth of 35 m. The thickness of this layer varies between 2 and 5 m. On the top overlies a permeable water-bearing layer in which the pollution plume is extended. The Netherlands Institute of Applied Geosciences (TNO) proposed the site.

The geophysical target is to map a pollution plume in the groundwater reservoir based on radio–MT data from four profiles with an average length of 400 m (Figure 7). The data along line 1 were selected for this discussion. Line 1 contains 42 radio–MT stations with 10-meter station spacing. The line starts about 100 m to the east of the eastern border of the waste dump.

**Robust estimation to identify nonplane-wave transmitters**

If all transmitters are in the far-field, i.e., the plane-wave assumption is valid, then the EM components strictly obey equations (1). Data measured from transmitters located close to the recording site obey different linear relations. If those near-field transmitters are few, it is possible to remove them by a robust technique involving the following steps.

![Fig. 6. Estimated Q-values in the (a) x- and (b) y-directions of the electric field. A damping factor equal to \( 10^{-10} \) times the largest eigenvalue was added to all eigenvalues.](image1)

![Fig. 7. Schematic of the location of radio–MT survey lines in Collendoorn, the Netherlands. Lines 1–3 are east–west directed, and line 4 is in the north–south direction.](image2)
First, using TSVD, the best model is found and the corresponding magnitude of the individual prediction errors is calculated by

\[ e_i = \left( \frac{(d_{obs}^i - d_{est}^i)^* \times (d_{obs}^i - d_{est}^i)}{\sigma_i^2} \right)^{1/2}, \]  

where \( * \) denotes the complex conjugate.

Denoting the spectral matrix for a particular transmitter frequency by \( S \), we find immediately for the magnitude of prediction error of, for example, \( E_x \)

\[ e(E_x) = \left\{ \frac{|Z_{xx}|^2 S_{H_x} H_x + |Z_{xy}|^2 S_{H_y} H_y + 2Re(Z_{xx}^* Z_{xy}^* S_{H_x} H_y)}{\sigma_{E_x}} \right\}^{1/2}. \]  

As we mentioned before, the background noise level is found as the median filtered power spectrum of each horizontal field. The choice of the width of filter to some extent affects the background level estimation. Figure 8a shows the estimated apparent resistivity and phase in the \( x \)-direction of the electric field at station 10 using both methods. Figure 8b represents the same quantities in the \( y \)-direction of the electric field. Results with the new method show less scatter in both directions. The resistivity variations are much more continuous and smooth compared to the standard band-averaging technique. This is because we assume that the transfer functions are constant in each subband when using the band-averaging method. In this case, the bandwidth is one octave. Especially at frequencies around 20 kHz where there are only a few transmitters, the distribution of frequencies in the subbands may become quite nonuniform. Referring to the borehole data provided by TNO, there are generally four layers, with the resistivity sequence resistive–conductive–resistive–conductive. As discussed in the introduction, this is one of the worst cases for using the band-averaging method. The irregular distribution of the radio transmitters (see Figure 1, e.g., in the frequency ranges between 25 and 37 kHz and between 85 and 110 kHz, which are completely void of transmitters) may also introduce some bias. All of these factors contribute to the scatter when using the band-averaging technique.

The most important difference between the estimates is that the new parametric method provides much smoother transfer functions and the standard errors of the estimates are more realistic because bias effects introduced in the band-averaging technique are largely reduced. Strictly speaking, the parametric technique still produces a small bias from the noise in the input channels \( H_x, H_y \). However, this bias is small and generally on the order of 1% using a typical minimum S/N ratio of 20 dB. The covariance and resolution matrices of the estimated \( Z_{xx} \) and \( Z_{xy} \), with \( p = 14 \) for station 10 are shown in Figures 9a and 9b. The largest covariances are found for the highest frequencies, where the best resolution is also obtained. The structure of the resolution matrix shows there is a little coupling between the diagonal \( (Z_{xx}) \) and off-diagonal \( (Z_{xy}) \) terms of the impedance tensor. The spread of the resolution kernels covers several neighboring frequencies, indicating the data do not uniquely determine the impedance tensor at the two-per-octave level used in this procedure. However, the resolution kernels at the highest frequencies are clearly more localized than they are at the lower frequencies. This is in good agreement with the fact that there are fewer available radio transmitters at lower frequencies.

Radio–MT data

The data frequencies range from 10 to 250 kHz (see Figure 1 as an example), and the model frequencies again are chosen to vary from 100 Hz to 25 MHz with a uniform logarithmic distribution and two model frequencies per octave. In the field, using north–south as the \( x \)-direction and east–west as the \( y \)-direction, the transfer functions are estimated with the band-averaging method with a bandwidth equal to one octave, such that there are 9 subbands with the fixed target frequencies. With the selected bandwidth, subbands overlap. For example, the first subband is 10–20 kHz, and the second one is 14.1–28.2 kHz. So they overlap from 14.1 up to 20 kHz. We applied the new technique to the data from line 1 at the Collendoorn waste dump. For comparison, the transfer functions are estimated at the same target frequencies as the band-averaging method. A detailed comparison for a single station and, subsequently, for the whole line is demonstrated.

Detailed analysis for a single station

The S/N ratio of the output channels is determined as the ratio between the peak values and the background noise level.
Fig. 8. (a) Estimated apparent resistivity and phase in the $x$-direction of the electric field. (b) Estimated apparent resistivity and phase in the $y$-direction of the electric field at station 10 on line 1, Collendoorn, the Netherlands. The dashed lines with circles represents the estimates and the errors with band averaging, and the full lines with stars show the parametric estimates and their errors. The number of eigenvectors was 16 for each direction, and the normalized $\chi^2$ was close to 4. No frequencies were deleted.
The apparent resistivities shown in Figures 10a and 10b using the two techniques are rather similar. The main difference occurs at the higher frequencies, where the parametric technique generally gives higher apparent resistivities. The corresponding phases are shown in Figures 10c and 10d. The main difference here is the smooth variation for the parametric case compared with the band-averaging technique. When we use the parametric approach and compare apparent resistivities and phases, it appears that the high apparent resistivities at the high frequencies are nicely consistent with the phases >45°.

Comparison of 1-D inversion results

The data shown in Figure 10 were inverted using the new least singular value inversion (LSVI) code of Pedersen and Bastani (unpublished manuscript). The results for both the band-averaged and the TSVD data are shown in Figure 11. A 2% error floor was imposed whereby nearly all data errors were increased to that level. The corresponding normalized data fits were calculated by

\[ Q = \frac{1}{2N} \sum_{i=1}^{N} \left( \frac{d_{i}^{\text{obs}} - d_{i}^{\text{est}}}{\sigma_{i}} \right)^{2}, \]

where \( d_{i}^{\text{obs}} \) and \( d_{i}^{\text{est}} \) are, respectively, measured and estimated apparent resistivities or phases at target frequency number \( i \). The data fits along line 1 in the \( x \)-direction of the electric field are shown in Figure 11a. Generally, the fits are equal to unity for the TSVD data and are noticeably smaller than the band-averaged data fits, which are on the order of 2 to 4. This improvement in the data fits is also reflected in the model sections of Figures 11b and 11c. Although the general structures are quite similar, the TSVD section shows a much better depth resolution. For example, at 20 to 30 m depth, the top of the resolved conductor that represents the pollution plume generated by the dump site is much better resolved by the TSVD section. The depth to the top of the plume is 25 m, measured in a borehole close to station 11—in very good agreement with the interpreted sections. The overall trend composed of four layers in the eastern to the central part of the sections is also confirmed by the borehole data.

CONCLUSIONS

The estimation of MT transfer functions from irregularly distributed sets of radio transmitters was cast into a linear inverse problem using a modified 1-D representation of the transfer functions. By this reformulation it becomes possible to estimate the complete impedance tensor or tipper vector using discrete frequencies irregularly distributed over the entire frequency band. These estimates are nearly bias free since the horizontal magnetic field power can easily be measured to within an error of 1%. Tests with synthetic data and a realistic distribution of radio transmitters show that the estimated transfer functions match the true transfer functions well to within their standard errors. Application of the method to a data set from Collendoorn, the Netherlands, shows that the apparent resistivities and phases reflect details in the subsurface that are confirmed by borehole logs. Comparing the band-averaging and parametric techniques clearly shows that the results from the latter are less scattered, are less affected by distribution...
of transmitter frequencies and azimuths, and generally have smaller standard errors.

REFERENCES
Fig. 11. Results of 1-D inversion of data in the x-direction of the electric field along line 1 in the dump site. (a) Estimated $Q$, where stars denote the estimated values using the parametric data and full circles show the same parameter using the band-averaged data. The model section obtained with (b) the band-averaged data, and (c) the parametric data.


